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Canonical second rank tensors for chiral SU(3) × SU(3)

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Abstract. The general second rank SU(3) tensor formed from a single hermitian octet vector is expressed in terms of a base set constructed to have particularly simple properties. The relevance to chiral lagrangian theories of hadron physics is discussed, and other applications are suggested.

1. Introduction

Recent developments in hadron physics (Gasiorowicz and Geffen 1969) have led to the problem of constructing the general second rank SU(3) tensor from a single hermitian octet vector, and to the need to form contractions of such tensors against one another. We shall briefly review how this specific problem arises, before giving a general solution and suggesting some alternative uses of the results.

One of the major techniques in studying the application of chiral algebras (both $K(2) = SU(2) \times SU(2)$ and $K(3) = SU(3) \times SU(3)$) to hadron dynamics (Gasiorowicz and Geffen 1969) has been the construction of explicit chiral invariant Lagrangians with their associated currents (Callan *et al* 1969), and with the subsequent addition of symmetry breaking terms. In order to construct such Lagrangians it is necessary to study the transformation of particle fields under the chiral group, and in particular to find the explicit form of transformation law for those fields forming nonlinear realizations. The general theory of such realizations (Coleman *et al* 1969 and Isham 1969) is well established but leads directly only to a power series in fields which is difficult to compute beyond low orders. For the case of the K(2) algebra there are two approaches which lead to general closed form expressions for the transformation laws. One of these is the algebraic approach of Weinberg (1968) and the other the matrix method associated with the name of Gürsey (Chang and Gürsey 1967). Both these methods are in principle capable of extension to the K(3) level. Macfarlane *et al* (1970, and references therein) have shown how the algebraic method leads either to equations of the sixth degree or to linked partial differential equations and have found some simple solutions, while in the alternative approach they discovered three particular models. More recently the main technical difficulty arising in the Gürsey method has been resolved by the explicit construction of the appropriate general three by three unitary matrix (Barnes *et al* 1972a). In either approach the second rank tensor which specifies the axial variation of the pseudoscalar meson fields (the Goldstone bosons of the scheme (Gasiorowicz and Geffen 1969)) is the primary objective, and for our present purposes we shall only indicate briefly how it enters in the algebraic approach where it appears most directly. The interested reader is urged to consult the references given above (particularly Weinberg 1968 and Macfarlane *et al* 1970) for greater detail.

There are sixteen generators of K(3) satisfying the usual commutation relations

$$
\begin{aligned}
[Q_i^V, Q_j^V] &= \mathrm{i}f_{ijk}Q_k^V \\
[Q_i^V, Q_j^A] &= \mathrm{i}f_{ijk}Q_k^A \qquad (i = 1\ldots 8) \\
[Q_i^A, Q_j^A] &= \mathrm{i}f_{ijk}Q_k^V
\end{aligned}
\tag{1}
$$

where f_{ijk} are the standard SU(3) structure constants (Gell-Mann and Ne'eman 1964), and the commutation relations

$$[Q_i^V, M_j] = \mathrm{i}f_{ijk}M_k \tag{2}$$

$$[Q_i^A, M_j] = \mathrm{i}F_{ij}(M) \tag{3}$$

define the transformation properties induced on the fields M_i used to describe the pseudoscalar mesons. The tensor F_{ij}, which contains two arbitrary functions of the two SU(3) invariants, is to be found (Macfarlane *et al* 1970) by writing the most general second rank tensor formed from the single octet M_i and imposing the Jacobi identity between a meson field and two axial generators. Once this is accomplished, the required transformation laws of all other fields may be found (and the required Lagrangians constructed) provided contractions of F_{ij} with itself and other second rank tensors (formed from M^i) can be calculated. Details of this development may be found in the work of Macfarlane *et al* (1970) where a list of ten independent second rank tensors is suggested; and Dittner (1971 private communication) has since shown that these ten tensors are indeed an independent and complete set. However the computational difficulties which arise when one works with this set are so formidable that a general solution has never been found by the algebraic method. Moreover the solutions obtained by the Gürsey method (Dondi 1971) give little insight and are intractable in use unless the properties of the general second rank tensors are understood. We now construct a base set of such second rank tensors, from a single octet, in a manner which makes subsequent computation quite straightforward.

2. The tensors

A general octet vector M^i is well known to specify a dual vector

$$N_i = d_{ijk}M_jM_k \tag{4}$$

and two independent SU(3) invariants

$$X = M_iM_i \tag{5}$$

and

$$Y = d_{ijk}M_iM_jM_k \tag{6}$$

where the constants d_{ijk} are taken as in the 'Eightfold Way' (Gell-Mann and Ne'eman 1964). These vectors have been extensively studied by Macfarlane *et al* (1970), and by Michel and Radicati (1968). The latter authors introduce particular vectors which are most important for our work. They call charge vectors those M^i for which $|Y|$ attains its maximum value of $(\frac{1}{3}X^3)^{1/2}$, and N^i is parallel to M^i. Special vectors are those M^i for which Y is zero, and N^i (which is then orthogonal to M^i) is itself a charge vector. We shall use the symbols q^i and s^i for hermitian charge and special vectors of unit norm. It has recently been established (Barnes *et al* 1972a) that a general hermitian vector M^i may be expressed in the form

$$M^i = \left(\frac{X}{3}\right)^{1/2} [N_i^+ \exp\{\mathrm{i}(\alpha+\tfrac{1}{2}\pi)\} + N_i^- \exp\{-\mathrm{i}(\alpha+\tfrac{1}{2}\pi)\}] \tag{7}$$

where

$$N_i^\pm = -\frac{\sqrt{3}}{2}(q_i \pm \mathrm{i}s_i) \tag{8}$$

$$\sin\alpha = \left(\frac{3Y^2}{X^3}\right)^{1/2} \tag{9}$$

and the special and charge vectors obey the relationships

$$s_is_i = 1 = q_iq_i \tag{10}$$

$$q_is_i = 0 \tag{11}$$

$$-\sqrt{3}\, d_{ijk}q_jq_k = q_i = \sqrt{3}\, d_{ijk}s_js_k \tag{12}$$

$$\sqrt{3}\, d_{ijk}s_jq_k = s_i \tag{13}$$

which make the subsequent algebra tractable. The inverses of equation (7) are easy to obtain using equations (10)–(13) and have been given explicitly (Barnes *et al* 1972a), thus q_i and s_i are known in terms of M^i and we may attempt to construct our base set of tensors from them. Now, as stated above, a set of ten tensors is required and we take initially

$$(F_s)_{ij} = \mathrm{i}f_{ikj}s_k \tag{14}$$

$$(F_q)_{ij} = \mathrm{i}f_{ikj}q_k \tag{15}$$

$$\sqrt{3}\, C_{ij} = \mathrm{i}d_{ipk}s_pf_{ktj}q_t \tag{16}$$

$$(\Sigma_2)_{ij} = \mathrm{i}(q_is_j - s_iq_j) \tag{17}$$

$$(\Sigma_3)_{ij} = q_is_j + s_iq_j \tag{18}$$

$$(\Sigma_1)_{ij} = q_iq_j - s_is_j \tag{19}$$

$$I_{ij} = q_iq_j + s_is_j \tag{20}$$

$$(D_s)_{ij} = \sqrt{3}\, d_{ikj}s_k \tag{21}$$

$$(D_q)_{ij} = \sqrt{3}\, d_{ikj}q_k \tag{22}$$

and the unit $(1)_{ij} = \delta_{ij}$, where the left hand sides suggest an obvious matrix notation which will now be employed to the exclusion of the indices. These matrices are all clearly hermitian, the first four being antisymmetric and the rest symmetric. We have taken this set by comparison with those of Macfarlane *et al* (1970), so that by substituting $(M^i \to s^i)$ and $(\sqrt{3}\, N^i \to q^i)$ into their work we have a check on many of the results we later derive. Moreover the work of Dittner (1971 and private communication) assures us that we have a complete and independent set of tensors, so that a general tensor may be written as a sum of products of functions of the two invariants with the members of our base set.

Since we are effectively dealing with eight by eight matrices it may be possible to find a set of eight P_α with the projective properties

$$P_\alpha P_\beta = \delta_{\alpha\beta} P_\beta \qquad \text{(no sum)} \tag{23}$$

$$\sum_{\alpha=1}^{8} P_\alpha = 1 \tag{24}$$

and if these can be identified with linear combinations of our provisional set and incorporated into our base set then subsequent manipulation clearly becomes simple. Our task becomes much easier when it is noticed that, because s^i and q^i are orthogonal unit vectors, the matrix

$$(1-I)_{ij} = \delta_{ij} - s_i s_j - q_i q_j \tag{25}$$

acts as a projector and allows us to treat the problem effectively as the direct sum of two by two and six by six problems.

The two by two part of the problem is clearly contained in the matrices defined by equations (17)–(20), and their multiplication rules

$$I\Sigma_a = \Sigma_a = \Sigma_a I \tag{26}$$

$$\Sigma_a \Sigma_b = I\delta_{ab} + \mathrm{i}\epsilon_{abc}\Sigma_c \tag{27}$$

where $(a, b = 1-3)$ and ϵ_{abc} is the Levi-Cività tensor, follow immediately from the orthonormality of the charge and special vectors. Obviously our notation now assumes some significance, for we have a simple SU(2) problem. Within this sector there are, of course, only two independent matrices with projective properties, and these may be taken to be

$$P_7 = \tfrac{1}{2}(I + n_a \Sigma_a) \tag{28}$$

and

$$P_8 = \tfrac{1}{2}(I - n_a \Sigma_a) \tag{29}$$

where n_a is an arbitrary SU(2) three component vector with unit norm.

There remains a six by six part of the problem defined by the six matrices

$$(1-I)F_s(1-I) = F_s \tag{30}$$

$$(1-I)F_q(1-I) = F_q \tag{31}$$

$$(1-I)C(1-I) = C \tag{32}$$

$$(1-I)D_s(1-I) = D_s - \Sigma_3 \tag{33}$$

$$(1-I)D_q(1-I) = D_q + \Sigma_1 \tag{34}$$

and $(1-I)$ which now plays the role of the unit matrix in this subspace. Clearly the remaining task is to exhibit six independent linear combinations of these matrices which have projective properties. For this purpose we need to compute all products of the matrices, and these results are given in the Appendix together with the well known SU(3) identities from which they follow (Macfarlane *et al* 1970). We have ordered the products so as to facilitate comparison with the limits of the corresponding equations in the Appendix of Macfarlane *et al* (1970), and in passing note agreement with all except (B20), (B22) and (B24) of that work. Macfarlane kindly sent us his corrections to the first two of these thus removing those discrepancies, and Dittner (private communication) has confirmed that an error exists in (B24), so that we now have a fair measure of confidence in our results. (Of course our equation (A.21) is much easier to derive than (B24).) With this machinery at our disposal we look for three projective matrices among the symmetric set, and a perusal of the Appendix soon reveals

$$P^0 = \tfrac{1}{3}\{(1-I)+2(D_q+\Sigma_1)\} \tag{35}$$

$$P^{\pm} = \tfrac{1}{3}\{(1-I)-(D_q+\Sigma_1)\pm\sqrt{3}(D_s-\Sigma_3)\} \tag{36}$$

as the appropriate ones. Taking traces we see that these define a separation of the space into the direct sum of three two by two subspaces, so that we expect to find one antisymmetric matrix in each of these new subspaces. Following our earlier procedure we define

$$A^0 = P^0F_sP^0 = F_s-2C \tag{37}$$

$$\sqrt{3}\,A^+ = 2P^+F_qP^+ = F_q+2\sqrt{3}\,C \tag{38}$$

$$\sqrt{3}\,A^- = 2P^-F_qP^- = F_q-2\sqrt{3}\,C \tag{39}$$

where we clearly have an independent set still, and the normalization has been taken for convenience. By using the results in the Appendix we see that

$$A^0A^0 = P^0 \tag{40}$$

$$A^+A^+ = P^+ \tag{41}$$

$$A^-A^- = P^- \tag{42}$$

while all other products are obvious from the properties of the projection operators. As the final step we may now introduce the notation

$$2P_1 = P_0+A_0 \tag{43}$$

$$2P_2 = P_0-A_0 \tag{44}$$

$$2P_3 = P^++A^+ \tag{45}$$

$$2P_4 = P^+-A^+ \tag{46}$$

$$2P_5 = P^-+A^- \tag{47}$$

$$2P_6 = P^--A^- \tag{48}$$

for these $P_i (i = 1\text{–}6)$ are manifestly matrices with the required projective properties. This gives our main result that the required general second rank tensor may be taken in the matrix form (with tensor indices suppressed)

$$F = \sum_{i=1}^{6} a_iP_i+a_II+\sum_{b=1}^{3} b_b\Sigma_b \tag{49}$$

where the a_i, b_b and a_I are functions of the SU(3) invariants X and Y, and where the two extra projection matrices defined in (28) and (29) should be employed whenever possible.

3. An example and conclusions

An immediate and important application of these results is, of course, to the specification of nonlinear realizations of K(3) and the construction of the corresponding chiral

Lagrangian as discussed above (Barnes *et al* 1972b). However our formalism also lends itself to a description of finite SU(3) transformations on an octet vector. This would clearly be of great importance in calculations of the type recently proposed by Dashen (1971), where finite (rather than infinitesimal) transformations are crucial. The orthogonal matrix required to transform an octet vector is

$$R = \exp(\mathrm{i}F_M) \tag{50}$$

where

$$(F_M)_{ij} = \mathrm{i}f_{ikj}M_k \tag{51}$$

and the M_j are now to be regarded as eight real parameters specifying the transformation. Substituting from equation (7) we find

$$R = \exp \mathrm{i}\sqrt{X}(F_q \sin\alpha + F_s \cos\alpha) \tag{52}$$

and hence

$$\begin{aligned} R = \exp \mathrm{i}\sqrt{X}\{&\tfrac{1}{2}\sqrt{3}(P_3 - P_4 + P_5 - P_6)\sin\alpha \\ &+\tfrac{1}{2}(2P_1 - 2P_2 + P_3 - P_4 - P_5 + P_6)\cos\alpha\} \end{aligned} \tag{53}$$

when expressed in terms of matrices with projective properties. If we note that the completeness is given by

$$\sum_{i=1}^{6} P_i + I = 1 \tag{54}$$

then

$$\begin{aligned} R = I &+ P_1 \exp(\mathrm{i}\sqrt{X}\cos\alpha) + P_2\exp(-\mathrm{i}\sqrt{X}\cos\alpha) \\ &+ P_3 \exp\{-\mathrm{i}\sqrt{X}\cos(\alpha+\tfrac{2}{3}\pi)\} + P_4\exp\{\mathrm{i}\sqrt{X}\cos(\alpha+\tfrac{2}{3}\pi)\} \\ &+ P_5 \exp\{\mathrm{i}\sqrt{X}\cos(\alpha-\tfrac{2}{3}\pi)\} + P_6\exp\{-\mathrm{i}\sqrt{X}\cos(\alpha-\tfrac{2}{3}\pi)\} \end{aligned} \tag{55}$$

follows at once. Setting α equal to zero we obtain the special case

$$\begin{aligned} R(M^i = \sqrt{X}\, s^i) = I &+ P_1\exp(\mathrm{i}\sqrt{X}) + P_2\exp(-\mathrm{i}\sqrt{X}) \\ &+ (P_3+P_6)\exp(\tfrac{1}{2}\mathrm{i}\sqrt{X}) + (P_4+P_5)\exp(-\tfrac{1}{2}\mathrm{i}\sqrt{X}) \end{aligned} \tag{56}$$

while when $\alpha = \frac{1}{2}\pi$ we find the charge case result

$$\begin{aligned} R(M^i = \sqrt{X}\, q^i) = (I + P_1 + P_2) &+ (P_3+P_5)\exp(\tfrac{1}{2}\mathrm{i}\sqrt{3X}) \\ &+ (P_4+P_6)\exp(-\tfrac{1}{2}\mathrm{i}\sqrt{3X}) \end{aligned} \tag{57}$$

where in each case the bracketed groups of matrices clearly have projective properties. Results equivalent to those in equations (50), (56) and (57) have recently been derived by Rosen (1971) who used as starting point the characteristic equation for the F_M matrix. Detailed comparison is difficult because his general answers are given in implicit form, but the general structures of our three results are identical and several terms checked at random also agree. We point out that there may be some additional advantage in our formalism for calculations of the Dashen type, since the arguments of Michel and

Radicati (1968) suggest that a smooth limit from the general case to those cases where charge and special vectors are involved may be crucial.

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Appendix

The SU(3) identities

$$f_{mli}f_{mjk}+f_{mlj}f_{mki}+f_{mlk}f_{mij} = 0 \tag{A.1}$$

$$f_{mli}d_{mjk}+f_{mlj}d_{mki}+f_{mlk}d_{mij} = 0 \tag{A.2}$$

$$3(d_{mli}d_{mjk}+d_{mlj}d_{mki}+d_{mlk}d_{mij}) = \delta_{li}\delta_{jk}+\delta_{lj}\delta_{ki}+\delta_{lk}\delta_{ij} \tag{A.3}$$

$$3f_{mij}f_{mkl} = 2(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+d_{mik}d_{mjl}-d_{mil}d_{mjk} \tag{A.4}$$

$$3d_{mij}d_{mkl} = \delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\delta_{ij}\delta_{kl}+f_{mik}f_{mjl}+f_{mil}f_{mjk} \tag{A.5}$$

are sufficient to allow calculation of products of the six matrices in the text. By simple contraction, and use of the above identities, we obtain the results

$$2D_sD_s = 1+I-\Sigma_1-D_q \tag{A.6}$$

$$2D_sD_q = \Sigma_3-D_s-2\mathrm{i}\Sigma_2 \tag{A.7}$$

$$2D_qD_s = \Sigma_3-D_s+2\mathrm{i}\Sigma_2 \tag{A.8}$$

$$2D_qD_q = 1+I+\Sigma_1+D_q \tag{A.9}$$

$$2F_sF_s = 1-I+\Sigma_1+D_q \tag{A.10}$$

$$2F_sF_q = D_s-\Sigma_3 = 2F_qF_s \tag{A.11}$$

$$2F_qF_q = 1-I-\Sigma_1-D_q \tag{A.12}$$

$$2D_sF_s = F_q = 2F_sD_s \tag{A.13}$$

$$2D_qF_q = -F_q = 2F_qD_q \tag{A.14}$$

$$D_sF_q \equiv 3C \equiv F_qD_s \tag{A.15}$$

$$D_qF_s = F_s-3C = F_sD_q \tag{A.16}$$

$$4CD_s = F_q = 4D_sC \tag{A.17}$$

$$12CF_s = 1-I-\Sigma_1-D_q = 12F_sC \tag{A.18}$$

$$2CD_q = -C = 2D_qC \tag{A.19}$$

$$4CF_q = D_s-\Sigma_3 = 4F_qC \tag{A.20}$$

$$24CC = 1-I-\Sigma_1-D_q \tag{A.21}$$

where matrix notation (as described in the text) has been used, and the order of the results has been arranged for easy comparison with Appendix B of Macfarlane *et al* (1970).

References

Barnes K J, Dondi P H and Sarkar S C 1972a *J. Phys.* A: *Gen. Phys.* **5** 555–62
—— 1972b *Nonlinear realizations of chiral algebras and phenomenological Lagrangians* (*Preprint*)
Callan C G, Coleman S, Wess J and Zumino B 1969 *Phys. Rev.* **177** 2247–50
Chang P and Gürsey F 1967 *Phys. Rev.* **164** 1752–62
Coleman S, Wess J and Zumino B 1969 *Phys. Rev.* **177** 2239–47
Dashen R 1971 *Phys. Rev.* D **3** 1879–90
Dittner P 1971 *PhD Thesis* University of London pp 40–73
Dondi P H 1971 *PhD Thesis* University of London pp 100–19
Gasiorowicz S and Geffen D A 1969 *Rev. mod. Phys.* **41** 531–74
Gell-Mann M and Ne'eman Y 1964 *The Eightfold Way* (New York: Benjamin) p 50
Isham C J 1969 *Nuovo Cim* A **59** 356–76
Macfarlane A J, Sudbery A and Weisz P 1970 *Proc. R. Soc.* A **314** 217–50
Michel L and Radicati L A 1968 *Symmetry Principles at High Energy* (New York: Benjamin) pp 19–48
Rosen S P 1971 *J. math. Phys.* **12** 673–81
Weinberg S 1968 *Phys. Rev.* **166** 1568–77